TRANSIENT FLOW IN AN OPEN DRY CHANNEL

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The case is that of planar flow on running water into a dry channel. There are several papers on this [1–7], in which either no allowance is made for resistance of the bed, or the form is appropriate to steady-state even flow, which is assumed to apply also to transient uneven flow. The flow is essentially of transient type in the present (dam-break) case.

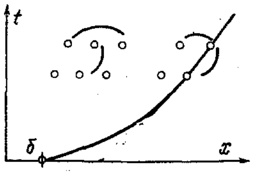

Fig. 1

The problem is here considered numerically, with allowance for the transient-state effects considered in [8].

1. APPROXIMATE FRICTIONAL STRESS AT THE BOTTOM NEAR THE FRONT

We use the equations for a planar open flow [8]

$$\frac{\partial u}{\partial t}+\frac{\partial u^2}{\partial x}+\frac{\partial uv}{\partial y}=g\left(\sin\alpha_0-\frac{\partial h}{\partial x}\cos\alpha_0\right)+$$
$$+\frac{1}{\rho}\frac{\partial \tau}{\partial y},\quad \frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0. \tag{1.1}$$

subject to the conditions

$$\tau=0,\quad \frac{\partial h}{\partial t}+u\frac{\partial h}{\partial x}=v \quad \text{for } y=h(t,x),$$
$$u=0,\quad v=0 \quad \text{for } y=0. \tag{1.2}$$

Here t is time, (x, y) is a cartesian coordinate system (x axis along the fixed rectilinear contour), u and v are the components of the velocity along x and y, p is pressure, and ρ and ν are the density and kinematic viscosity respectively [8].

Function τ is found from additional considerations; in particular, τ is a known function of t, x, and y in §1.

Conditions (1.2) are for integration of (1.1) with respect to y from 0 to h.

Here we assume that h(t, x) and u(t, x, y) may be discontinuous. The problem then has to be considered by replacing the differential equations by integral ones, which is done as follows.

Let $x=\xi(t)$ be some line in the (x, t) plane. We perform the following transformations of the independent variables: $\vartheta_1=x-\xi(t)$, $\vartheta_2=t$, $\vartheta_3=y$. Then the first equation of (1.1), with (1.2), may be put as

$$\int_\gamma A_1 d\vartheta_1+A_2 d\vartheta_2=\int_0^{\vartheta_2}\int_{-\vartheta_1}^{\vartheta_1}\left(gh\sin\alpha_0-\frac{\tau_0}{\rho}\right)d\vartheta_1 d\vartheta_2,$$

$$A_1=-\int_0^h u\,d\vartheta_3,$$

$$A_2=-\xi^{\cdot}\int_0^h u\,d\vartheta_3+\int_0^h u^2 d\vartheta_3+g\frac{h^2}{2}\cos\alpha_0,\quad \xi^{\cdot}=\frac{d\xi(\vartheta_2)}{d\vartheta_2}. \tag{1.3}$$

Here τ_0 is the stress on the bed, γ is the contour of the region, $0\le\vartheta_{2*}\le\vartheta_2$, $-\vartheta_1\le\vartheta_{1*}\le\vartheta_1$. Let functions u and h be sufficiently smooth everywhere except on the line $\vartheta_1=0$ in the plane of ϑ_1 and ϑ_2, while on that line they have a discontinuity of the first kind.

Then differentiation of (1.3) with respect to ϑ_2 gives

$$A_2\Big|_{-\vartheta_1}^{\vartheta_1}=\int_{-\vartheta_1}^{\vartheta_1}\left(gh\sin\alpha_0-\frac{\tau_0}{\rho}\right)d\vartheta_1+\vartheta_1\frac{\partial M_1}{\partial\vartheta_2}. \tag{1.4}$$

Here M_1 is the sum of the mean values of the integral $A_1(\vartheta_{1*},\vartheta_2)$ in the ranges $-\vartheta_1\le\vartheta_{1*}\le 0$, $0\le\vartheta_{1*}\le\vartheta_1$. We assume the restrictions

$$|u|<\infty\quad(0\leqslant h<\infty),\qquad\left|\frac{\partial M_1}{\partial\vartheta_2}\right|<\infty\quad(0\leqslant\xi^{\cdot}<\infty)$$

and introduce the symbol

$$\lim_{\vartheta_1\to 0}\int_{-\vartheta_1}^{\vartheta_1}-\frac{\tau_0}{\rho}d\vartheta_1=C(\vartheta_2). \tag{1.5}$$

Then Eq. (1.4) with $\vartheta_1\to 0$ gives

$$\int_0^{h_-}u_-^2 d\vartheta_3-\xi^{\cdot}\int_0^{h_-}u_- d\vartheta_3+g\frac{h_-^2}{2}\cos\alpha_0=$$
$$=\int_0^{h_+}u_+^2 d\vartheta_3-\xi^{\cdot}\int_0^{h_+}u_- d\vartheta_3+g\frac{h_-^2}{2}\cos\alpha_0+C. \tag{1.6}$$

Here u_+, h_+ and u_-, h_- are the limiting values of u and h as ϑ_1 tends to zero from left and right respectively.

Similarly, we get from the second equations in (1.1) and (1.2) that

$$\int_0^{h_-}u_- d\vartheta_3-\xi^{\cdot}h_-=\int_0^{h_+}u_+ d\vartheta_3-\xi^{\cdot}h_+. \tag{1.7}$$

Here we impose the restriction $|\partial M_2/\partial\vartheta_2|<\infty$. Here M_2 is the sum of the mean values of $h(\vartheta_{1*},\vartheta_2)$ in the intervals $-\vartheta_1\le\vartheta_{1*}\le\le 0$, $0\le\vartheta_{1*}\le\vartheta_1$. To get motion of the water along the dry bed we put $h_+=0$ in Eq. (1.7). Then

$$\int_0^{h_-}u_-^2 d\vartheta_3-\frac{1}{h_-}\left(\int_0^{h_-}u_- d\vartheta_3\right)^2=-g\frac{h_-^2}{2}\cos\alpha_0+C. \tag{1.8}$$

From Gelder's inequality

$$\frac{1}{h_-}\left(\int_0^{h_-}u_- d\vartheta_3\right)^2\leqslant\int_0^{h_-}u_-^2 d\vartheta_3$$

so from (1.8) we have

$$C\geqslant {}^1/_2\, gh_-^2\cos\alpha_0\qquad(\cos\alpha_0\equiv\text{const}>0). \tag{1.9}$$

Let τ_0 be such that $C\equiv 0$; then Eq. (1.9) implies that $h_-\equiv 0$.

Let h be such that $h_-\equiv 0$; then Eq. (1.8) implies that τ_0 must be such that $C\equiv 0$. It is thus necessary and sufficient to have $C\equiv 0$ in order to get $h_-\equiv 0$; if $C>0$, then $h_->0$. In fact, $h_-=0$ for some value $\vartheta_2=\vartheta_{20}$; then (1.8) gives $C(\vartheta_{20})=0$, which conflicts with the condition $C>0$, which means that $h_->0$.

Relation (1.5) with $C\equiv 0$ implies some restriction on τ_0. For instance, this restriction is obeyed if

$$\frac{\tau_0}{\rho}=K(\vartheta_2)\,\vartheta_{10}^{-\sigma}\quad(\vartheta_{10}\geqslant 0),\qquad\frac{\tau_0}{\rho}\equiv 0\quad(\vartheta_{10}\leqslant 0) \tag{1.10}$$

near $\vartheta_1=0$. Here the real number $\sigma<1$, $K(\vartheta_2)$ is bounded, and $\vartheta_{10}==-\vartheta_1$.

The stress at the bottom $\tau_0 = 0$ for $\sigma < 0$ and $\vartheta_{10} = +0$. An exception occurs when $0 \leq \sigma < 1$.

We thus obtain a conception of the frictional stress on the bottom of an open flow for a small region around $\vartheta_1 = 0$.

2. COEFFICIENT OF FRICTIONAL RESISTANCE

In deducing the resistance coefficient λ it was assumed [8] that the depth $h \gg k$, in which k is the average height of the roughness projections. Here the formula for λ is extended to the case where h is nearly zero, or with $h = 0$ at isolated points. Such values of h occur, for example, near the front of a wave moving along a dry bed. For uniform motion we have [8]

$$\lambda_p = \left[\alpha \Big/ \left(\ln \frac{h}{k} + \alpha\beta - 1\right)\right]^2. \qquad (2.1)$$

Here α and β are universal turbulence constants. Formula (2.1) shows that the monotonic increase in λ_p ceases for $h \leqslant k$.

If $k \ll h$, as was assumed in deducing λ_p and λ, we modify the formula for λ_p in such a way that it increases monotonically as h decreases; λ is also altered. The resulting λ_p and λ are taken as valid for all h (the calculated results are in good agreement with experiment). As

$$\ln \frac{h}{k} + \alpha\beta - 1 = \ln \left[1 + e^{\alpha\beta-1} \frac{h - ke^{1-\alpha\beta}}{k}\right]$$

and $k \ll h$, we may put

$$\ln \frac{h}{k} + \alpha\beta - 1 = \ln \left[1 + \frac{h}{k} e^{\alpha\beta-1}\right]$$

neglecting $k\exp(1 - \alpha\beta)$ as small relative to h. Then we see that for uniform motion we have

$$\lambda_p = [\alpha / \ln (1 + h / D)]^2.$$

Also, λ for transient flow becomes

$$\lambda = \left[\alpha \frac{1 + \sqrt{1 + \omega}}{2\omega_1}\right]^2,$$

$$\left(\omega_1 = \ln\left(1 + \frac{h}{D}\right) + 0.5, \quad D = ke^{1-\alpha\beta}\right)$$

$$\omega = 2g\omega_1 h \, \frac{\sin \alpha_0 - \cos \alpha_0 \, \partial h / \partial x}{\alpha^2 \, | w | \, w}, \qquad w = \frac{1}{h} \int_0^h u dy. \qquad (2.2)$$

Here w is the water speed. We assume that (2.2) applies for h small.

3. ENTRY OF WATER INTO A DRY CHANNEL

Consider a horizontal channel of rectangular cross-section extending to infinity in both directions and with a thin partition at $x = 0$; initially, there is a water depth H = constant for $x < 0$, the

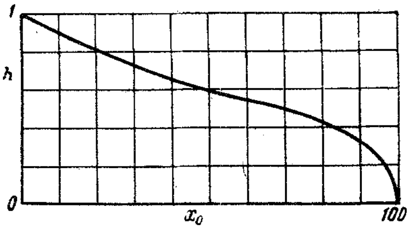

Fig. 2

water being at rest, with no water on the other side. The partition fails instantaneously at $t = 0$, the problem being to determine the motion for all x and subsequent t (dam-breaking) [1]. Let $\alpha_1 > 1$ be such that

$$\alpha_1 w^2 = \frac{1}{h} \int_0^h u^2 dy.$$

It is unusual to take the second correction α_1 as equal to one for transient-state flow in open channels. Then integration of (1.1) subject to (1.2) with $\alpha_0 = 0$ along y, from 0 to h, gives

$$\frac{\partial wh}{\partial t} + \frac{\partial w^2 h}{\partial x} + gh \frac{\partial h}{\partial x} = -\frac{\tau_0}{\rho}, \qquad \frac{\partial h}{\partial t} + \frac{\partial wh}{\partial x} = 0. \qquad (3.1)$$

Replacement of τ_0/ρ by $\lambda|w|w$ [8] and the transformation

$$w = w^* \sqrt{gH}, \quad h = h^* H, \quad t = t^* \sqrt{H/g}, \quad x = x^* H$$

with omission of the asterisk, gives

$$\frac{\partial w}{\partial t} + w \frac{\partial w}{\partial x} + \frac{\partial h}{\partial x} = -\frac{\lambda}{h} | w | w,$$
$$\frac{\partial h}{\partial t} + w \frac{\partial h}{\partial x} + h \frac{\partial w}{\partial x} = 0. \qquad (3.2)$$

The characteristics of the system of (3.2) are

$$\frac{dx}{dt} = w \pm \sqrt{(1 - \mu) h} \qquad \left(\mu = \frac{1}{\alpha}\left(\frac{\lambda}{1 + \omega}\right) 0.5\right).$$

The system of (3.2) is nonlinear. In the (x, t) plane we distinguish a region Γ bounded by the lines

$$\frac{dx}{dt} = -\sqrt{1 - \mu_*} = -c, \qquad x(0) = 0,$$
$$\left(\mu_* = \frac{1}{2\ln(1 + HD^{-1}) + 1}\right) \qquad (3.3)$$
$$dx / dt = w_*, \qquad x(0) = \delta. \qquad (3.4)$$

Here w_* is the velocity of the water at the point where the free surface meets the bed (the speed of the wave front). We denote Γ for $t \geq 0$ by Γ_+ and specify the following boundary conditions at the

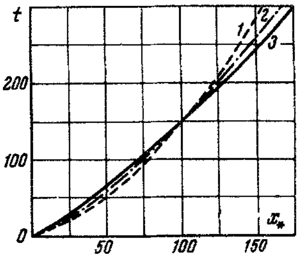

Fig. 3

edges: $w = 0$ in (3.3), $h = 0$ in (3.4), and $w = 0$ and $h = h(x)$ on the part $0 \leq x \leq \delta$, with $h(\delta) = 0$. We have to solve this problem for (3.2) in region Γ_+.

It is readily shown that Eqs. (3.3) and (3.4) are the characteristics of (3.2) for the solution satisfying the conditions

1) in (3.3)

$$\partial h / \partial x < 0$$

2) in (3.4)

$$| w | < \infty, \quad | \partial w/\partial t | < \infty, \quad | \partial w/\partial x | < \infty.$$

4. NUMERICAL METHOD AND RESULTS

We perform the transformations $x^* = x + ct$, $t^* = t$ in (3.2) and omit the asterisk; subsequently by (3.2) we understand the transformed system describing the motion of the water in the moving coordinate system. In this new system, (3.3) becomes $x = 0$ while (3.4) becomes $dx/dt = w_* + c$, $x(0) = \delta$.

The following are some features encountered in solving this system by this method.

The calculations are performed for $0 < \delta \ll 1$ (this means that the front is nearly vertical).

We consider the following initial forms for the free surface:

$$\text{(a)} \ h(x) = 1 - \frac{x}{\delta} \qquad (0 \leqslant x \leqslant \delta),$$

$$\text{(b)} \ h(x) = \left(1 - \frac{x}{\delta}\right)^{(1-\sigma)/2} \qquad \left(\begin{matrix} 0 \leqslant x \leqslant \delta \\ 0 \leqslant \sigma < 1 \end{matrix}\right).$$

In case (b) the initial depth takes into account the singularity of the solution of (3.1) with Eq. (1.10) near the front of a continuous wave [4]. The calculations show that there is no substantial difference between cases (a) and (b).

At any time t the number N of points along the x axis is constant, i. e., the step Δx_i changes with t. The elements of the moving difference net in the (x, t) plane and the boundary curve $dx/dt = w_* + c$, $x(0) = \delta$ are shown in Fig. 1.

The system of (3.2) is approximated as follows. The derivatives are

$$\frac{\partial \psi}{\partial t} \approx \frac{\psi_n^{i+1} - \psi_n^i}{\Delta t_{i+1}} - \frac{x_n^{i+1} - x_n^i}{\Delta t_{i+1}} \frac{\psi_{n+1}^{i+1} - \psi_{n-1}^{i+1}}{2\Delta x_{i+1}},$$

$$\frac{\partial \psi}{\partial x} \approx \frac{\psi_{n+1}^{i+1} - \psi_{n-1}^{i+1}}{2\Delta x_{i+1}},$$

the coefficients for the derivatives on the left being taken on the previous layer; the right part of the first equation in the system is

$$\lambda \frac{\operatorname{sign}(w)}{h} w^2 \approx \lambda_n^i \frac{\operatorname{sign}(w_n^i)}{h_n^i} [2w_n^i w_n^{i+1} - (w_n^i)^2].$$

This approximation for the right part is used in order to obtain a stable difference system.

The system of (3.2) has a singularity at $dx/dt = w_* + c$, $x(0) = \delta$, because $h = 0$. In calculating $w_{N-1}^{i+1} = w_{N-1}^{i+1}\, w_N^{i+1}$ in (3.2), the coefficients to the derivatives and the expression for λ/h are taken at the point $x_{N-1}^i = (N - 1)\Delta x_i$, while the derivatives are approximated as

$$\frac{\partial \psi}{\partial t} \approx \frac{\psi_N^{i+1} - \psi_N^i}{\Delta t_{i+1}} - \frac{x_N^{i+1} - x_N^i}{\Delta t_{i+1}} \frac{\psi_N^{i+1} - \varphi_{N-1}^{i+1}}{\Delta x_{i+1}},$$

$$\frac{\partial \psi}{\partial x} \approx \frac{\psi_N^{i+1} - \psi_{N-1}^{i+1}}{\Delta x_{i+1}}.$$

This system of difference equations is solved by matrix methods; the system has not been examined for stability, but the results show that the system is stable.

The following are some results. The parameters were taken as

$$H = 0.11 \text{ m}, \quad T = 300 \ (0 \leqslant t \leqslant T); \quad k = 0.0028 \text{ m}.$$

This choice of parameters is in accordance with experiment [4].

The solution is obtained in dimensionless form. Figure 2 shows h as a function of x_0 at $t = T$, with x_0 determined from $x = 4.64x_0$. The curves in Fig. 3 are

$$x_* = \int_0^t w_* dt$$

which characterize the motion of the front; the theoretical and experimental curves of [4] are denoted by 1 and 2 respectively, while curve 3 is from the present calculation.

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